

# NEW FAMILIES OF SCALING MULTIPARTICLE DISTRIBUTIONS

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Recently equations for the generating functional in the perturbative quantum chromodynamics (QCD) have been extended by including the non-perturbative dissipation in QCD jets. The resulting equations have been solved *rigorously* and new family of scaling solutions , the so-called  $\delta$  - scaling, generalizing the well-known Kubo-Nielsen-Olesen scaling law for hadron multiplicity distributions have been found. The relevance of  $\delta$ - scaling is discussed in the Landau - Ginzburg theory of phase transitions. Preliminary application of these ideas to the  $p\bar{p}$  data of the UA5 Collaboration is presented.

## 1 Introduction

A suitable framework for the comparison of multiplicity distributions at different energies is provided by the idea of KNO scaling<sup>1</sup>, which states that at sufficiently high energies :  $\lim_{s \rightarrow \infty} (\langle n \rangle \sigma_n) (\sum_n \sigma_n)^{-1} = f(z)$ , where  $\sigma_n$  is the cross-section for producing exactly  $n$  particles,  $\langle n \rangle$  is the average multiplicity of produced particles and  $f(z)$  is an energy-independent scaling function in  $z = n/\langle n \rangle$ . The KNO scaling is fulfilled in  $e^+e^-$  collisions up to the highest energies considered<sup>2</sup> . The  $e^+e^-$  data are well accounted for by the Perturbative Quantum Chromodynamics (PQCD)<sup>3</sup> calculations and, in particular, by the Fragmentation - Inactivation - Binary (FIB) process<sup>4,5</sup>, which is equivalent to PQCD with dissipation effects in jet cascading<sup>6</sup> . In  $p\bar{p}$  collisions, KNO scaling holds up to ISR energies but, at higher energies, UA5 Collaboration reported significant breaking of the KNO scaling<sup>8</sup> .

The KNO scaling was derived from the hypothesis of Feynman scaling<sup>1</sup>. It is now known that Feynman scaling is violated at very high energies so other arguments should be looked for. In this context, a possible relation between the KNO scaling and the phase transition in Feynman-Wilson gas<sup>7</sup> as well as the criticality of self-similar FIB process<sup>4,5</sup> was pointed out. It seems therefore that KNO scaling may have more profound reasons to appear. Since

new data on multiplicity distributions at still higher energies will be available soon, we believe that the discussion of information contained in multiplicity distributions is particularly urgent and challenging.

## 2 Order parameter fluctuations

Let us consider a self-similar statistical system of size  $N$ , in which phase changes are characterized by the order parameter  $\eta$ . Self-similarity in such system means that for different scales which are characterized by different values of order parameter  $\langle \eta \rangle$ , fluctuations of normalized order parameter  $\eta / \langle \eta \rangle$  are *identical*. We know that this is a feature of thermodynamical systems such as the Ising model at the critical point of second order phase transition. Defining the anomalous dimension as :

$$g = \lim_{N \rightarrow \infty} \frac{d}{d \ln N} (\ln \langle N|\eta| \rangle) \quad , \quad (1)$$

one can show that the partition function :  $Z_N \sim N^{-(1-g)} \sim \langle |\eta| \rangle$  and the probability density  $P[\eta]$  obeys the scaling law :

$$\langle |\eta| \rangle P[\eta] = \Phi(z_{(1)}) = \Phi \left( \frac{\eta - \eta^*}{\langle |\eta| \rangle} \right) \quad , \quad (2)$$

in the centered variable  $z_{(1)} = (\eta - \eta^*) / \langle |\eta| \rangle$ , with  $\eta^*$  the most probable value of  $\eta$  for given parameters of the system. We call (2) *the first scaling law*. The scaling limit is defined by the asymptotic behaviour of  $P[\eta]$  when  $\eta \rightarrow \infty$ ,  $\langle |\eta| \rangle \rightarrow \infty$ , but  $z_{(1)}$  has a finite value. The first scaling law holds also for fluctuations of any power  $\eta^\zeta$  ( $\zeta > 0$ ) of the order parameter<sup>9</sup>. If the order parameter is related to cluster multiplicity, like in the FIB process<sup>5,9</sup>, and if the most probable value of the order parameter is close enough to its average value, then (2) can be written in an equivalent form to the KNO scaling<sup>1</sup>. For large  $z_{(1)}$ , the scaling function behaves as :

$$\Phi(z_{(1)}) \sim \exp \left( -z_{(1)}^{1/(1-g)} \right) \quad , \quad (3)$$

allowing for an alternative determination of anomalous dimension. For equilibrium systems at the critical point of second-order phase transition,  $g$  must be contained in between 1/2 and 1.

We have to ask, what happens if the observable quantity is not an order parameter but the  $N$ -dependent function of it like :  $m = N^\kappa - N\eta$ , where  $\kappa > g$ . For large  $N$ ,  $|m|$  is of order  $N^\kappa$ . Writing (2) with  $m$  instead of  $\eta$  and

Figure 1: The multiplicity distribution in FIB model for  $\alpha = -1/3$  with the Gaussian inactivation (6) ( $\beta = 0, c = 1, \sigma = 1$ ) in the scaling variables  $\delta = 0.76$  are plotted for systems of different sizes :  $N = 2^{10}$  (circles),  $2^{12}$  (squares),  $2^{14}$  (crosses).  $10^7$  events have been calculated. Details of the calculation can be found in Ref. 11.

taking into account :  $P[\eta]d\eta = P[m]dm$  , one finds the generalized law :

$$\langle |m| \rangle^\delta P[m] = \Phi(z_{(\delta)}) \equiv \Phi\left(\frac{m - m^*}{\langle |m| \rangle^\delta}\right) , \quad \delta = \frac{g}{\kappa} < 1 \quad (4)$$

, which will be called in the following *the  $\delta$  - scaling*. The scaling function  $\Phi(z_{(\delta)})$  depends only on the scaled variable :  $z_{(\delta)} = (m - m^*) / (\langle |m| \rangle^\delta)$  . The scaling function  $\Phi(z_{(\delta)})$  in (4) has *identical form* as  $\Phi(z_{(1)})$  except for the inversion of abscissa axe. In particular, its tail for large  $z_{(\delta)}$  has the same form :

$$\Phi(z_{(\delta)}) \sim \exp\left(-z_{(\delta)}^{1/(1-g)}\right) . \quad (5)$$

$\delta = 1/2$  is a particular case when  $\Phi(z_{(\delta)})$  is nearly Gaussian<sup>10</sup> and  $\langle m \rangle \sim N$ . This limit, called *the second - scaling*, has been found outside of the transition line in the shattering phase<sup>11</sup>. More about the  $\delta$  - scaling interested reader will find in Ref. 9.

The dynamical realization of  $\delta$  - scaling has been found in FIB model with finite-scale dissipation effects. For an appropriate choice of the fragmentation

kernel function, FIB is exactly equivalent to PQCD with the dissipation in jet fragmentation. In Fig. 1 we show the cluster multiplicity distributions calculated in FIB model for different initial sizes and plotted in the scaling variable  $z_{(0.76)}$ . The calculation has been done for homogeneous fragmentation function :  $F_{j,N-j} \sim [j(N-j)]^\alpha$  and the dissipation function which at small scales is approximated by the Gaussian inactivation rate function :

$$I_k = ck^\beta \exp[-[(k-1)^2/N^2]/(2\sigma^2)]. \quad (6)$$

The parameters of fragmentation and inactivation kernel functions are :  $\alpha = -1/3$ ,  $\beta = 0$ ,  $c = 1$ ,  $\sigma = 1$ . Of course, the inactivation becomes scale-invariant if  $\sigma \rightarrow \infty$  and for this choice of  $\alpha$  and  $\beta$ , FIB process is in the shattering phase where the second - scaling holds. On the other hand, if  $\sigma \rightarrow 0$ , then  $I_k \rightarrow 0$  and the fragmentation process becomes independent of the cluster size  $k$ . This limiting situation happens at the transition line where the first-scaling holds. Thus,  $\delta$  - scaling in FIB appears rather as a cross-over phenomenon between first- and second- scaling domains than the phenomenon of gradually increasing deviation between the order parameter in the studied process and the relevant observable. Below, we shall address this question rigorously in the Landau-Ginzburg (L-G) theory of phase transitions.

### 3 $\delta$ scaling in the Landau-Ginzburg theory

Let us consider the homogeneous L-G free energy density as the simplest example of second-order phase transition. If  $\epsilon$  is the relative distance to the critical point, then the free-energy density is :  $f(\eta) = \epsilon\eta^2 + b\eta^4 + \dots$ , with  $b$  a positive constant. Under this form, the most probable value of  $\eta$  is implicitly set to 0 in the disordered phase. For finite size systems, it is often more convenient to work with the extensive order parameter :  $\hat{m} = N\eta$ . Keeping the first two terms in  $f(\eta)$ , one may show that the probability that the system will be in a state  $\hat{m}$  for a given  $\epsilon$  is :

$$P[\hat{m}] = \frac{1}{Z_N} \exp\left(-\epsilon \frac{\hat{m}^2}{N} - b \frac{\hat{m}^4}{N^3}\right) \quad , \quad (7)$$

where  $Z_N$  is defined by the normalization of  $P[\hat{m}]$ . Without loss of generality, we consider now the case when  $\eta$  is contained between 0 and 1, i.e.,  $\hat{m} > 0$ . We know that at the transition point ( $\epsilon = 0$ ), the function  $P[\hat{m}]$  verifies the first-scaling<sup>4,9</sup>, while below the critical point ( $\epsilon < 0$ ),  $P[\hat{m}]$  verifies the second-scaling<sup>10</sup>. These results are valid in the thermodynamic limit. How the finite system moves on between these two scalings when the control parameter  $\epsilon$

moves away from 0? To answer this question, let us first write down the average value of  $\hat{m}$  :

$$\langle \hat{m} \rangle = N^{3/4} \frac{\int_0^{N^{1/2}} u^{1/2} \exp(-\epsilon N^{1/2}u - bu^2) du}{\int_0^{N^{1/2}} u^{-1/2} \exp(-\epsilon N^{1/2}u - bu^2) du} . \quad (8)$$

When  $N \rightarrow \infty$ , then the term in brackets is only a function of  $\epsilon N^{1/2}$ . This is the finite-size scaling of this system near its critical point. It can be rewritten formally as :

$$\langle \hat{m} \rangle = N^{3/4} \psi(\epsilon N^{1/2}) \quad (9)$$

At the transition point,  $\psi$  has a constant value  $\psi(0)$ , and  $\langle \hat{m} \rangle$  behaves as a power of  $N$  with the mean-field exponent  $3/4$ . At the critical point, this exponent is identical to the anomalous dimension defined in (1). In the following discussion, it will be denoted by  $g$  both for systems at the critical state and outside of it. In the whole ordered phase, we have  $g = 1$ . Eq. (8) shows also that the cross-over between critical and non-critical phases must be discussed according to the values of  $\epsilon N^{1/2}$ . Moreover, since we have to recover the non-critical case :  $\langle \hat{m} \rangle \propto N$ , for finite negative  $\epsilon$  therefore :  $\psi(x) \sim (-x)^{1/2}$ , when  $x$  is large and negative. Let us note first the difference between the average value of  $\hat{m}$ , as given by (8), and its most probable value (for  $\epsilon \leq 0$ ) :  $\hat{m}^* = \pm N^{3/4} [(-\epsilon N^{1/2})/2b]^{1/2}$ . For  $\epsilon < 0$ , one can approximate  $P[\hat{m}]$  by a Gaussian function :

$$\frac{N^{3/2}}{\langle \hat{m} \rangle} P[\hat{m}] \sim \sqrt{\frac{4b}{\pi}} \exp\left(\frac{\epsilon}{N}(\hat{m} - \hat{m}^*)^2\right) . \quad (10)$$

Suppose now that the order of magnitude of  $\epsilon$  is such that  $\epsilon \simeq 1/N^a$ , with some positive exponent  $a$ , smaller than  $1/2$  to ensure that  $\epsilon N^{1/2}$  is a large number for large  $N$ . Then,  $\langle \hat{m} \rangle$  and  $\hat{m}^*$  are both of similar order of magnitude. This means that the prefactor  $(N^{3/2}/\langle \hat{m} \rangle)$  in (10) is of order :  $\langle \hat{m} \rangle^{(1+a)/(2-a)}$ . In a similar way, the quantity  $(N/\epsilon)$  appearing in (10) is of order :  $\langle \hat{m} \rangle^{2(1+a)/(2-a)}$ . Therefore, one can now write eq. (10) in the  $\delta$ -scaling form (4). Obviously, we recover the known particular cases :  $\delta = 1$  for  $a = 1/2$ , and  $\delta = 1/2$  for  $a = 0$ . Here, however, the  $\delta$ -scaling appears as a finite-size effect and the tail of scaling function for large arguments has *always* Gaussian form, in contrast to the  $\delta$ -scaling due to  $N$ -dependent change of order parameter which was discussed in the preceding section.

We have seen above that  $\delta$ -scaling appears either as a result of incompatibility between the observable and the order parameter, or as a cross-over

Figure 2: The UA5 multiplicity distributions in  $p\bar{p}$  reactions<sup>8</sup> for three different energies : 200 GeV (circles), 546 GeV (squares) and 900 GeV (diamonds), are plotted in the  $\delta$  - scaling variables for  $\delta = 0.9$ .

between first- and second-scaling. Both  $N$ -dependent phenomena may actually be related one to another. In general, it is not so easy to conclude which of the two cases prevails without an additional information, and this point needs some comments here. Up to now, we did not need to know explicitly the total size  $N$  of the system, because the  $\delta$ -scaling is not expressed explicitly in terms of  $N$ . In the practical applications, we have to know only that the data corresponds to a *constant value* of  $N$ . Nevertheless, we could have in principle access to the  $N$  - dependence of average quantities such as  $\langle m \rangle$ . It should be :

$$\langle m \rangle \sim N^{g/\delta} \quad , \quad (11)$$

for the change of variable as in Sect. 2 and :

$$\langle m \rangle \sim N^{2g/(\delta+1)} \quad , \quad (12)$$

for the finite-size cross-over effect described above in the mean-field approximation. In the latter case,  $g = 3/4$ . If one knows from the experimental data the value of scaling parameter  $\delta$  and the  $N$ -dependence of  $\langle m \rangle$ , then one may find the value of the anomalous dimension if the cause of  $\delta$  - scaling can

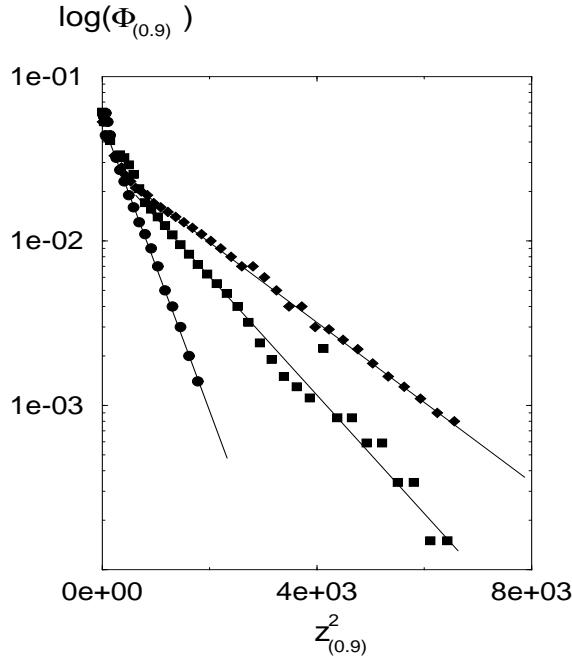


Figure 3: The logarithm of scaled multiplicity distributions at UA5 data<sup>8</sup> and three different energies : 200 GeV (circles), 546 GeV (squares) and 900 GeV (diamonds), plotted versus  $z_{(0.9)}^2 \equiv ((m - m^*) / \langle m \rangle^{0.9})^2$  for  $z_{(0.9)} > 0$ .

be decided by analyzing the tail of the scaling distribution. Below, we shall illustrate this on the example of UA5 data.

#### 4 Application and conclusions

We shall apply now the concepts developed above to the  $p\bar{p}$  data of UA5 Collaboration<sup>8</sup> which show significant deviations with respect to the KNO (first-) scaling. If one plots data for different  $\sqrt{s}$  in KNO variables, one notice that the maxima of  $\langle m \rangle P[m]$  function are shifted, the height of maximum increases with  $\sqrt{s}$  and the width decreases. Distributions at  $\sqrt{s} = 200, 546$  and 900 GeV are plotted in Fig. 2 in the  $\delta$  - scaling variables for  $\delta = 0.9$ . Even if uncertainty on the average multiplicity is probably large, the scaling looks well. As noticed before, we can find argument for the cause of  $\delta$  - scaling by analyzing the tail of scaling function. This is shown in Fig. 3 where the logarithm of scaling distribution is plotted versus  $z_{(0.9)}^2$ . We see that the three curves tend to be linear in this plot, proving that tails of  $\Phi(z_{(\delta)})$  are essentially Gaussian. This provides a rather convincing argument that the fragmenting system is predominantly in the ordered phase ( $\delta < 1$ , and Gaussian tail), close to a critical point ( $\delta \simeq 1$ ). Moreover, a plot (unfortunately, for only three

values of  $\sqrt{s}$ ) of  $\langle m \rangle$  vs  $\sqrt{s}$  shows that :  $\langle m \rangle \sim (\sqrt{s})^{0.35}$ . Using relation (12), allows then to write :

$$N^g \sim (\sqrt{s})^{0.33} , \quad (13)$$

what should be the proper scaling of the order parameter with the size of the system. Of course, one should be aware of the preliminary character of this extracted value of  $g$ . First of all, data in full space are not numerous. Secondly, the analysis is global in the sense that it concerns different classes of events mixed together. Nevertheless, the  $\delta$  - scaling analysis discussed in this work may become the powerful tool, allowing for an intelligent analysis of multiplicity data at different energies, which could give access to the determination of the anomalous dimension in the particle production at ultrarelativistic energies.

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$f(z)$

3.0

2.0

1.0

0.0

0.0

0.5

1.0

1.5

2.0

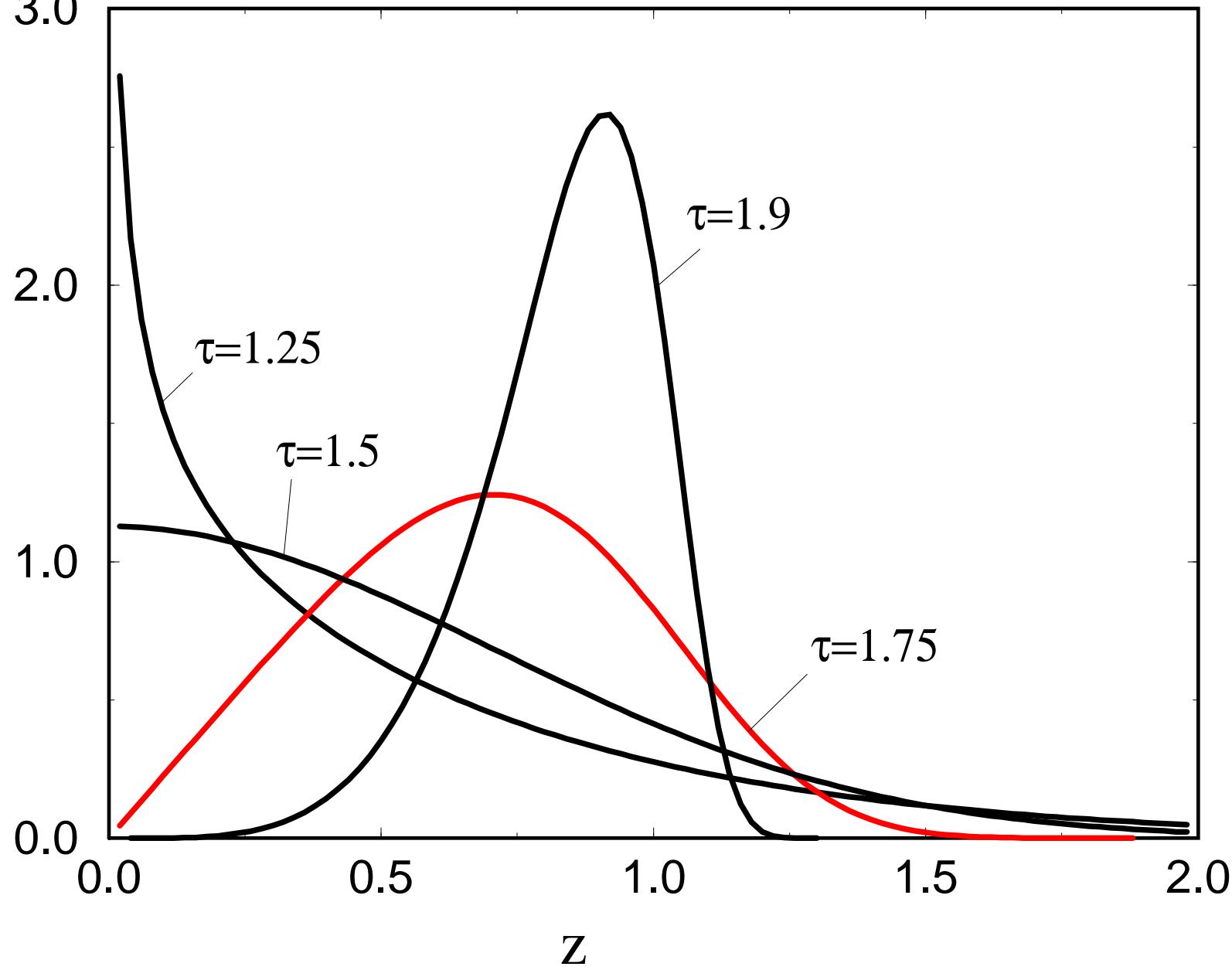
$z$

$\tau=1.25$

$\tau=1.5$

$\tau=1.9$

$\tau=1.75$



$$\langle m \rangle^{1/2} P(m)$$

